

Miyazawa Multiplier Model in disaggregated Household income Group

Abstract

The important work of Miyazawa (1976) on endogenizing households in an input–output model generates various multiplier matrices.¹⁰ A comprehensive overview of the explicit demographic-economic interactions in the Miyazawa structure and its applications can be found in the collection of papers in Hewings *et al.* (1999). In this section we depart from some of the notation used elsewhere in this book, in order to be consistent with that used by Miyazawa, since virtually all subsequent discussion and application of the Miyazawa framework has continued to use his notation. Specifically, this means that we will now define $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$ (instead of \mathbf{L} , since Miyazawa uses \mathbf{L} for another purpose, as we will see below).

Disaggregated Household Income Groups

We assume that households can be separated into q distinct income-bracket groups and that payments by producers to wage earners in each of those groups can be identified. Let $\mathbf{V}_{(q \times n)} = [v_{gj}]$, where v_{gj} represents income paid to a wage earner in income bracket g ($g = 1, \dots, q$) per dollar's worth of output of sector j . This is a generalization (to q rows) of the single row of household input coefficients or labor input coefficients in Chapter 2, $\mathbf{h}_R = [a_{n+1,1}, \dots, a_{n+1,n}]$. Similarly, let $\mathbf{C}_{(n \times q)} = [c_{ih}]$, where c_{ih} is the amount of sector i 's product consumed per dollar of income of households in income group h ($h = 1, \dots, q$); this is a generalization (to q columns) of the single

column of household consumption coefficients in Chapter 2, $\mathbf{h}_C = \begin{bmatrix} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix}$, and yet

another use for \mathbf{C} in input-output discussions. So the augmented matrix of coefficients is

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{(n \times n)} & \mathbf{C}_{(n \times q)} \\ \mathbf{V}_{(q \times n)} & \mathbf{0}_{(q \times q)} \end{bmatrix}, \text{ and the expanded input-output system is}$$

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{f}^* \\ \mathbf{g} \end{bmatrix} \quad (6.37)$$

where $\mathbf{y}_{(q \times 1)}$ is a vector of total income for each of the income groups, $\mathbf{f}^*_{(n \times 1)}$ is a vector of final demands excluding household consumption (now endogenized) and $\mathbf{g}_{(q \times 1)}$ is a vector of exogenous income (if any) for the income groups.

Assume that $\mathbf{g}_{(q \times 1)} = \mathbf{0}$; then the two matrix equations in the system in (6.37) are

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{y} + \mathbf{f}^* \text{ and } \mathbf{y} = \mathbf{V}\mathbf{x} \quad (6.38)$$

From (6.37)

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{C} \\ -\mathbf{V} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \quad (6.39)$$

Using results on inverses of partitioned matrices (Appendix A) it is not difficult to show that the elements of the partitioned inverse in (6.39) can be expressed as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{B}[\mathbf{I} + \mathbf{C}(\mathbf{I} - \mathbf{VBC})^{-1}\mathbf{VB}] & \mathbf{BC}(\mathbf{I} - \mathbf{VBC})^{-1} \\ (\mathbf{I} - \mathbf{VBC})^{-1}\mathbf{VB} & (\mathbf{I} - \mathbf{VBC})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \quad (6.40)$$

where, as noted, $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$.

This can be simplified if, following Miyazawa, we define $\mathbf{VBC} = \mathbf{L}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1} = (\mathbf{I} - \mathbf{VBC})^{-1}$, so that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{B}(\mathbf{I} + \mathbf{CKVB})_{(n \times n)} & \mathbf{BCK}_{(n \times q)} \\ -\mathbf{KVB}_{(q \times n)} & \mathbf{K}_{(q \times q)} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \quad (6.41)$$

Miyazawa defines \mathbf{L} as the matrix of "inter-income-group coefficients" and \mathbf{K} as the "interrelational income multiplier" matrix. A typical element of \mathbf{L} is $l_{gh} = v_{gi}b_{ij}c_{jh}$; this shows the direct increase in the income of group g resulting from expenditure of

an additional unit of income by group h . Reading from right to left, household demand (expenditure) of c_{jh} by group h for the output of sector j requires $b_{ij}c_{jh}$ in output from sector i and this, in turn, means income payments from sector i in the amount of $v_{gi}b_{ij}c_{jh}$ to households in group g . Similarly, each element in $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$ indicates the total increase (direct, indirect and induced) in the income of one group that results from expenditure of an additional unit of income by another group. (An illustration of this approach can be found in the matrix of interrelational income multipliers, \mathbf{K} , for 11 income groups in the USA for 1987 that is shown in Rose and Li, 1999.)

From (6.41),

$$\mathbf{x} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* \quad (6.42)$$

and

$$\mathbf{y} = \mathbf{KVBf}^* \quad (6.43)$$

In (6.42), the effect of final demands on outputs is seen to be the product of two distinct matrices. The first is the Leontief inverse of the open model, \mathbf{B} . The second is $(\mathbf{I} + \mathbf{CKVB})$; this augments the final demand stimulus, \mathbf{If}^* , by \mathbf{CKVBf}^* , which endogenizes the total income spending effect. Again, starting at the right, \mathbf{Bf}^* generates the initial output (without household spending), \mathbf{VBf}^* indicates the resultant initial income payments to each group, \mathbf{KVBf}^* multiplies that into total income received in each group – this is exactly what is described by the result in (6.43) – and, finally, \mathbf{CKVBf}^* translates that received income into consumption (demand) by each group on each sector's output. Miyazawa denotes \mathbf{KVB} the “multi-sector income multiplier” matrix (or the “matrix multiplier of income formation”), indicating the direct, indirect and induced incomes for each income group generated by the initial final demand.

6.4.2 Miyazawa's Derivation

Miyazawa first derives the results on the interrelational multiplier matrix without reference to partitioned matrices [in Miyazawa, 1976, Chapter 1, sections II(2)–III(1); the partitioned inverse structure appears later in Chapter 1, section III(3)]. He makes extensive use of partitioned matrices later in the book – especially in Part 2 on internal and external matrix multipliers. This is a direction that has been explored and expanded considerably in much of the work of Sonis, Hewings and others (summarized in Sonis and Hewings, 1999, which also contains an extensive set of references to their work). A second direction of research that extends the input–output framework to incorporate interactions between economic and demographic components is associated with the many publications of Batey, Madden and others (summarized in Batey and Madden, 1999, again with many references).

We present Miyazawa's initial approach here primarily for completeness, and because the results are often discussed (briefly) in this form in the literature. He begins with

$$\mathbf{x} = \mathbf{Ax} + \mathbf{CVx} + \mathbf{f}^*$$

from (6.38). From this,

$$\mathbf{x} = (\mathbf{I} - \mathbf{A} - \mathbf{CV})^{-1} \mathbf{f}^* \quad (6.44)$$

and with $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$, straightforward matrix algebra gives

$$(\mathbf{I} - \mathbf{A} - \mathbf{CV}) = (\mathbf{B}^{-1} - \mathbf{CV})\mathbf{B}\mathbf{B}^{-1} = (\mathbf{I} - \mathbf{CVB})\mathbf{B}^{-1}$$

Substituting into (6.44),

$$\mathbf{x} = [(\mathbf{I} - \mathbf{CVB})\mathbf{B}^{-1}]^{-1} \mathbf{f}^*$$

and, from the rule for inverses of products,

$$\mathbf{x} = \mathbf{B}(\mathbf{I} - \mathbf{CVB})^{-1} \mathbf{f}^* \quad (6.45)$$

In this form, we find the original Leontief inverse, \mathbf{B} , postmultiplied by $(\mathbf{I} - \mathbf{CVB})^{-1}$ which Miyazawa termed the “subjoined inverse matrix.”

A further variation is possible and is sometimes used. Starting with (6.45) and, as earlier, with $\mathbf{VBC} = \mathbf{L}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$, then

$$\mathbf{K}(\mathbf{I} - \mathbf{VBC}) = \mathbf{I}$$

Premultiply both sides by \mathbf{C} and postmultiply both sides by \mathbf{VB} ,

$$\mathbf{CK}(\mathbf{I} - \mathbf{VBC})\mathbf{VB} = \mathbf{CVB} \text{ or } \mathbf{CK}(\mathbf{VB} - \mathbf{VBCVB}) = \mathbf{CVB}$$

Factor out \mathbf{VB} to the left and then subtract both sides from \mathbf{I} , giving

$$\mathbf{I} - \mathbf{CKVB}(\mathbf{I} - \mathbf{CVB}) = \mathbf{I} - \mathbf{CVB} \text{ or } \mathbf{I} = \mathbf{CKVB}(\mathbf{I} - \mathbf{CVB}) + \mathbf{I} - \mathbf{CVB}$$

Regrouping terms

$$\mathbf{I} = (\mathbf{I} + \mathbf{CKVB})(\mathbf{I} - \mathbf{CVB})$$

and so, from the fundamental definition of an inverse,

$$(\mathbf{I} - \mathbf{CVB})^{-1} = (\mathbf{I} + \mathbf{CKVB})$$

Putting this result into (6.45) gives

$$\mathbf{x} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* \quad (6.46)$$

as in (6.42).

Miyazawa suggests that if labor input coefficients, in \mathbf{V} , and household consumption coefficients, in \mathbf{C} , are less stable than interindustry coefficients (in \mathbf{A} and consequently in \mathbf{B}), there is an advantage to using the format in (6.46) instead of (6.45). Namely, a revised subjoined inverse, $(\mathbf{I} - \mathbf{CVB})^{-1}$, whose order is n , can be found by using \mathbf{K} , whose order is q “... [which] in most cases is very much smaller than n ...” (Miyazawa, 1976, p. 7). However, inverting large matrices is no longer the concern that it was in the 1970s.

From (6.46), household income, $y = Vx$, is seen to be

$$y = VB(I + CKVB)f^* = (I + VBCK)VBf^* = (I + LK)VBf^*$$

But since $K = (I - L)^{-1}$, $(I - L)K = I$, $LK = K - I$, so $(I + LK) = K$, and

$$y = KVBf^* \quad (6.47)$$

as in (6.43).

6.4.3 Numerical Example

We expand the numerical example from Chapter 2, assuming a three-sector economy with households divided into two income groups. Let the augmented coefficients matrix be

$$\bar{A} = \begin{bmatrix} A & C \\ V & 0 \end{bmatrix} = \begin{bmatrix} 0.15 & 0.25 & 0.05 & 0.1 & 0.05 \\ 0.2 & 0.05 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.25 & 0.05 & 0.01 & 0.1 \\ 0.05 & 0.1 & 0.08 & 0 & 0 \\ 0.12 & 0.05 & 0.1 & 0 & 0 \end{bmatrix}$$

In particular, labor income coefficients for the two household groups are given in the two rows of $V = \begin{bmatrix} 0.05 & 0.1 & 0.08 \\ 0.12 & 0.05 & 0.1 \end{bmatrix}$, and consumption coefficients for those same two

groups are given in the two columns of $C = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.1 \\ 0.01 & 0.1 \end{bmatrix}$

Given V , C , and $B = (I - A)^{-1} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$, the relevant

Miyazawa matrices are easily found to be

$$VBC = \begin{bmatrix} .0574 & .0454 \\ .0601 & .0480 \end{bmatrix} \text{ and } K = (I - VBC)^{-1} = \begin{bmatrix} 1.0642 & .0507 \\ .0671 & 1.0536 \end{bmatrix}$$

For example, in this illustration, a direct increase of \$1 in income to households in group 1 leads to a 6.7 cent (k_{21}) increase in income payments to households in group 2. Similarly,

$$KVB = \begin{bmatrix} .1898 & .2162 & .1960 \\ .2716 & .1894 & .2106 \end{bmatrix}$$

In this case, for example, an additional unit of final demand for the goods of sector 1 generates 27.16 cents in new income for group 2. Furthermore,

$$B(I - CVB)^{-1} = \begin{bmatrix} 1.4445 & .4994 & .3234 \\ .6496 & 1.4609 & .7062 \\ .6577 & .5644 & 1.3648 \end{bmatrix} \text{ and } BCK = \begin{bmatrix} .2476 & .1545 \\ .3642 & .2492 \\ .1923 & .2258 \end{bmatrix}$$

(The reader can make appropriate interpretations of the elements in each of these matrices.)

In this case, the Leontief inverse for the augmented system can easily be found directly; it is¹¹

$$(\mathbf{I} - \bar{\mathbf{A}})^{-1} = \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_{11} & \bar{\mathbf{B}}_{12} \\ \bar{\mathbf{B}}_{21} & \bar{\mathbf{B}}_{22} \end{bmatrix} = \begin{bmatrix} 1.4445 & .4994 & .3234 & .2476 & .1545 \\ .6496 & 1.4609 & .7062 & .3642 & .2492 \\ .6577 & .5644 & 1.3648 & .1923 & .2258 \\ .1898 & .2162 & .1960 & 1.0642 & .0507 \\ .2716 & .1894 & .2106 & .0671 & 1.0536 \end{bmatrix}$$

and the correspondences with elements in $\bar{\mathbf{B}}$ are exactly as expected, namely $\mathbf{K} = \bar{\mathbf{B}}_{22}$, $\mathbf{KVB} = \bar{\mathbf{B}}_{21}$, $\mathbf{BCK} = \bar{\mathbf{B}}_{12}$ and $\mathbf{B}(\mathbf{I} - \mathbf{CVB})^{-1} = \bar{\mathbf{B}}_{11}$.

6.4.4 Adding a Spatial Dimension

We saw in Chapter 3 that interregional or multiregional input-output models were conveniently represented in partitioned matrix form. To incorporate the Miyazawa structure into an IRIO- or MRIO-style model, assume that we have p regions ($k, l = 1, \dots, p$) with n sectors ($i, j = 1, \dots, n$) each and that we have identified q household income groups ($g, h = 1, \dots, q$) in each region. Then the augmented \mathbf{A} matrix would be

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} \begin{matrix} (np \times np) & (np \times pq) \\ (pq \times np) & (pq \times pq) \end{matrix}$$

where

$$\mathbf{A}_{(np \times np)} = \begin{bmatrix} \mathbf{A}^{11} & \dots & \mathbf{A}^{1p} \\ \vdots & \ddots & \vdots \\ \mathbf{A}^{p1} & \dots & \mathbf{A}^{pp} \end{bmatrix} \begin{matrix} (n \times n) & & (n \times n) \\ & \ddots & \\ (n \times n) & & (n \times n) \end{matrix} = [a_{ij}^{kl}], \quad \mathbf{C}_{(np \times pq)} = \begin{bmatrix} \mathbf{C}^{11} & \dots & \mathbf{C}^{p1} \\ \vdots & \ddots & \vdots \\ \mathbf{C}^{p1} & \dots & \mathbf{C}^{pp} \end{bmatrix} \begin{matrix} (n \times q) & & (n \times q) \\ & \ddots & \\ (n \times q) & & (n \times q) \end{matrix} = [c_{ih}^{kl}],$$

and

$$\mathbf{V}_{(pq \times np)} = \begin{bmatrix} \mathbf{V}^{11} & \dots & \mathbf{V}^{1p} \\ \vdots & \ddots & \vdots \\ \mathbf{V}^{p1} & \dots & \mathbf{V}^{pp} \end{bmatrix} \begin{matrix} (q \times n) & & (q \times n) \\ & \ddots & \\ (q \times n) & & (q \times n) \end{matrix} = [v_{gj}^{kl}].$$

Notice that consumption coefficients require knowledge of the spending habits of consumers in each income group in each region on goods from each sector in each region. Similarly, the labor input coefficients require knowledge on payments to laborers in each income group in each region by each sector in each region.

Table 6.8 Interrelational Interregional Income Multipliers

Region of Income Receipt	Region of Income Origin				Row Total
	1	2	3	4	
1	1.23	0.12	0.16	0.07	1.57
2	0.11	1.28	0.13	0.05	1.57
3	0.11	0.03	1.06	0.01	1.14
4	0.44	0.56	0.50	1.77	3.28
Column Total	1.81	1.99	1.85	1.90	

Source: Hewings, Okuyama and Sonis, 2001, Table 9.

The elements in the partitioned inverse in (6.41) will have the same dimensions as $\bar{\mathbf{A}}$, namely

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{B}(\mathbf{I} + \mathbf{CKVB}) & \mathbf{BCK} \\ \mathbf{KVB} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix}$$

$(np \times np)$ $(np \times pq)$
 $(pq \times np)$ $(pq \times pq)$

Clearly, this is potentially very demanding of data. However, an illustrative application can be found in Hewings, Okuyama and Sonis (2001) for a 53-sector, four-region model (Chicago and three surrounding suburbs), without division into income groups – that is, $n = 53$, $p = 4$, and $q = 1$. In this case the income formation impacts are across regions rather than income groups. In particular, \mathbf{K} is a 4×4 matrix; it is shown in Table 6.8.¹²

Reading down column 1 for illustration, we find that from an increase of \$1 in income in Region 1, an additional \$0.23 is generated in Region 1, \$0.11 in Regions 2 and 3, and \$0.44 in Region 4. Column sums have an interpretation similar to the more usual output multipliers; they indicate the new income generated throughout the four-region system (Chicago metropolitan area) of an additional \$1 in income in the region at the top of the column. Row sums are a measure of additional income in each region at the left as a result of a \$1 income increase in each region. (As with row sums of the usual Leontief inverse, these are generally less useful results than the column sums.) Often, results in empirically derived interrelational multiplier matrices are normalized in some way to account, for example, for differences in sizes of the regions being studied. A complete interregional Miyazawa analysis would require that we distinguish several income brackets in each region (that is, $q > 1$) and then create consumption coefficients and labor input coefficients for each of those brackets (in each region).

6.5 Gross and Net Multipliers in Input–Output Models

6.5.1 Introduction

Leontief's earliest formulations (for the USA in 1919, 1929, and 1939) were in terms of “net” accounts. The fundamental balance equations had no z_{ii} or a_{ii} terms; in the empirical tables the on-diagonal elements were zero.

[The interindustry transactions table] would naturally have many empty squares. Those lying along the main diagonal are necessarily left open because our accounting principle does not allow for registration of any transaction within the same firm ...” (Leontief, 1951, p. 13)

The output of an industry ... is defined with exclusion of the products consumed by the same industry in which they have been produced. Thus $a_{11} = a_{22} = \dots = a_{ii} = \dots = a_{mm} = 0$ by definition. (Leontief, 1951, p. 189)

The 1947 US input–output tables discussed and published in Evans and Hoffenberg (1952) include on-diagonal transactions, coefficients, and inverse elements; in that sense these tables are “gross.” They point out that the inverse figures can be adjusted to exclude intra-sector transactions but they do not suggest that as a preferable alternative.¹³ In Leontief *et al.* (1953, Chapter 2 by Leontief) the equations in the text are gross but the tables and the equations in the Mathematical Note to Chapter 2 are net. In virtually all later publications (for example, Leontief, 1966, Chapters 2 and 7) on-diagonal elements are included.¹⁴ (For a thoughtful discussion of net and gross input–output accounts, see Jensen, 1978.) This net/gross distinction led to the concept of input–output “net” multipliers, which we explore below.

6.5.2 Multipliers in the Net Input–Output Model

We consider only square systems. Generating a net model simply means that the main diagonals of \mathbf{Z} and \mathbf{A} contain only zeros, and that the gross output vector is reduced by the amount of each sector's intraindustry transactions. As usual, denote by $\hat{\mathbf{Z}}$ the diagonal matrix containing the elements z_{ii} . Then let $\mathbf{Z}_{net} = \mathbf{Z} - \hat{\mathbf{Z}}$, and $\hat{\mathbf{x}}_{net} = \hat{\mathbf{x}} - \hat{\mathbf{Z}}$; this latter is a diagonal matrix of sectoral outputs in the *net* system from which on-diagonal (intrasectoral) transactions have been removed.¹⁵ As usual, input coefficients are found for the net system as

$$\mathbf{A}_{net} = \mathbf{Z}_{net}(\hat{\mathbf{x}}_{net})^{-1} = (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

and

$$(\mathbf{I} - \mathbf{A}_{net}) = \mathbf{I} - (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

¹³ In contrast, Georgescu-Roegen (1971) argues that diagonal elements in an input–output model (“internal flows”) must be suppressed.

¹⁴ Early input–output tables in the UK (for example, for 1954 and 1963) were presented in “net” form (UK, Central Statistical Office, 1961 and 1970). Fifteen-sector versions of these tables appear in Allen and Lecomber (1975) and Barker (1975).

¹⁵ Alternative notation uses $\check{\mathbf{Z}}$ instead of \mathbf{Z}_{net} , and similarly for \mathbf{A}_{net} and \mathbf{x}_{net} . We avoid that convention because it becomes cumbersome when the vector \mathbf{x}_{net} needs a hat to indicate the associated diagonal matrix – and a “^” on top of a “v” is just too much.

We now examine an alternative expression for the right-hand side. [This demonstration appears to have originated in Weber, 1998 (in German). It is apparently not widely known, at least outside the German-speaking world.] Using the observation that $(\hat{\mathbf{x}} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = \mathbf{I}$, it can be shown that¹⁶

$$(\mathbf{I} - \mathbf{A}_{net}) = [(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

Taking the inverse of both sides,

$$\mathbf{L}_{net} = (\mathbf{I} - \mathbf{A}_{net})^{-1} = \{[(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}\}^{-1}$$

and using the matrix algebra rule for inverses of products (for appropriately sized matrices) that $(\mathbf{MNP})^{-1} = \mathbf{P}^{-1}\mathbf{N}^{-1}\mathbf{M}^{-1}$,

$$\mathbf{L}_{net} = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})\hat{\mathbf{x}}^{-1}(\mathbf{I} - \mathbf{A})^{-1} = \hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L} \quad (6.48)$$

from which

$$(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} = \hat{\mathbf{x}}^{-1}\mathbf{L} \quad (6.49)$$

[Notice from (6.48) that $\mathbf{L}_{net} = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})\hat{\mathbf{x}}^{-1}\mathbf{L} = (\mathbf{I} - \hat{\mathbf{A}})\mathbf{L}$, where $\hat{\mathbf{A}} = \hat{\mathbf{Z}}\hat{\mathbf{x}}^{-1}$.]¹⁷

Consider household income multipliers for the two systems. Given a vector of total household income by sector, $\mathbf{z}_h = [z_{n+1,1}, \dots, z_{n+1,n}]$, then $\mathbf{h} = \mathbf{z}_h\hat{\mathbf{x}}^{-1}$ and $\mathbf{h}_{net} = \mathbf{z}_h(\hat{\mathbf{x}}_{net})^{-1}$ are the vectors of earnings coefficients in the gross and net systems, respectively. From (6.49),

$$\mathbf{z}_h(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} = \mathbf{z}_h\hat{\mathbf{x}}^{-1}\mathbf{L}$$

or

$$\mathbf{h}_{net}\mathbf{L}_{net} = \mathbf{h}\mathbf{L}$$

Thus, the income multipliers in the two systems are equal, and therefore for studies in which these kinds of multiplier results are of interest, it makes no difference which model is used.

This result is equally valid for most other multipliers – value-added, household income, pollution-generation, energy use, etc. – associated with productive activity (Table 6.4). The only exception is for output multipliers – $\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$ and $\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net}$; they will not be equal,¹⁸ since from (6.48) $\mathbf{L}_{net} = \hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L}$. However, the transformation from one to the other is straightforward, namely

$$\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net} = \mathbf{i}'\hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L}$$

¹⁶ This particular expression for the identity matrix may seem unmotivated, but it cleverly allows for a significant rewriting of the expression for $(\mathbf{I} - \mathbf{A}_{net})$. For the interested reader, the derivation is:

$(\mathbf{I} - \mathbf{A}_{net}) = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} - (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\hat{\mathbf{x}} - \hat{\mathbf{Z}}) - (\mathbf{Z} - \hat{\mathbf{Z}})](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = (\hat{\mathbf{x}} - \mathbf{Z})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\mathbf{I} - \mathbf{Z}\hat{\mathbf{x}}^{-1})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$.

¹⁷ This fact was noted by Evans and Hoffenberg (1952, p. 140) who used a verbal argument and not a matrix algebra demonstration.

¹⁸ Except for the trivial and uninteresting case when $\mathbf{x} = \mathbf{x}_{net}$.

or

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \mathbf{i}'\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net}$$

(Recall that order of multiplication of diagonal matrices makes no difference.)

*Numerical Example*¹⁹ Let $\mathbf{Z} = \begin{bmatrix} 150 & 500 & 50 \\ 200 & 100 & 400 \\ 300 & 500 & 50 \end{bmatrix}$ so $\mathbf{Z}_{net} = \mathbf{Z} - \hat{\mathbf{Z}} =$

$$\begin{bmatrix} 0 & 500 & 50 \\ 200 & 0 & 400 \\ 300 & 500 & 0 \end{bmatrix} \text{ If } \mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} .15 & .25 & .05 \\ .2 & .05 & .4 \\ .3 & .25 & .05 \end{bmatrix};$$

$$\hat{\mathbf{x}}_{net} = \begin{bmatrix} 850 \\ 1900 \\ 950 \end{bmatrix}, \mathbf{A}_{net} = \mathbf{Z}_{net}(\hat{\mathbf{x}}_{net})^{-1} = \begin{bmatrix} 0 & .2632 & .0526 \\ .2353 & 0 & .4211 \\ .3529 & .2632 & 0 \end{bmatrix}$$

Then $\mathbf{L} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$ and $\mathbf{L}_{net} = (\mathbf{I} - \mathbf{A}_{net})^{-1} = \begin{bmatrix} 1.1603 & .3615 & .2133 \\ .5010 & 1.2807 & .5656 \\ .5414 & .4646 & 1.2241 \end{bmatrix}$

In this case,

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = [2.4623 \quad 2.2624 \quad 2.1348]$$

$$\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net} = [2.2026 \quad 2.1067 \quad 2.0030]$$

Here $\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1} = \begin{bmatrix} 1.1765 & 0 & 0 \\ 0 & 1.0526 & 0 \\ 0 & 0 & 1.0526 \end{bmatrix}$ and so $\mathbf{m}(o) = \mathbf{i}'\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} =$

$$[1.1765 \quad 1.0526 \quad 1.0526] \begin{bmatrix} 1.1603 & .3615 & .2133 \\ .5010 & 1.2807 & .5656 \\ .5414 & .4646 & 1.2241 \end{bmatrix} = [2.4623 \quad 2.2624 \quad 2.1348]$$

as expected.

Finally, let $\mathbf{z}_h = [100, 120, 80]$ (household income payments); then

$$\mathbf{h} = [0 \quad .10 \quad 0.06 \quad 0.08] \text{ and } \mathbf{h}_{net} = [0 \quad .1176 \quad 0.0632 \quad 0.0842]$$

from which

$$\mathbf{hL} = \mathbf{h}_{net}\mathbf{L}_{net} = [0.2137 \quad 0.1625 \quad 0.1639]$$

again as expected.

6.5.3 Additional Multiplier Variants

(Indirect Effects)/(Direct Effects) A number of analysts have taken the view that multipliers should not include the initial stimulus, as they do when the basic

¹⁹ We use the 3×3 example from earlier but now disregard the fact that sector 3 is households and simply treat this as a general three-sector model illustration.

definition is “total effects”/“direct effects.” For example, for output multipliers this means the \$1 of new final-demand for sector j which turns into \$1 of new sector j output. The usual resolution is simply to subtract 1 from each of the elements in $\mathbf{m}(o)$. This is equivalent to replacing \mathbf{L} by $(\mathbf{L} - \mathbf{I})$ in the formula for $\mathbf{m}(o)$, since $\mathbf{i}'(\mathbf{L} - \mathbf{I}) = \mathbf{i}'\mathbf{L} - \mathbf{i}'\mathbf{I} = \mathbf{m}(o) - \mathbf{i}'$. (For example, see Oosterhaven, Piek and Stelder, 1986.)²⁰ Of course this will not change the *rankings* of the sectors, but it certainly has implications for other kinds of calculations in which the multipliers are used.

The same adjustment [subtracting 1 or using $(\mathbf{L} - \mathbf{I})$] is appropriate for any Type I or Type II multiplier (Table 6.3). As an example, when $\mathbf{r} = \mathbf{h}$, the Type I multiplier, $\mathbf{m}(h) = \mathbf{h}\mathbf{L}$ would be converted to $\mathbf{h}(\mathbf{L} - \mathbf{I})\hat{\mathbf{h}}^{-1} = \mathbf{h}\mathbf{L}\hat{\mathbf{h}}^{-1} - \mathbf{h}\mathbf{I}\hat{\mathbf{h}}^{-1} = \mathbf{m}(h) - \mathbf{i}'$.

“Growth Equalized” Multipliers Policy makers may wish to know the impact on a particular sector of a general expansion in final demand in all sectors (for example, to help identify “bottlenecks”) or of changing patterns of final demand. One approach involves what have been called “growth-equalized” multipliers. (See, for example, Gray *et al.*, 1979, and Gowdy, 1991, for these and many additional multipliers.) The motivation is clear: “... size variation among economic sectors prevents meaningful comparisons of multipliers ... to add \$1 of output to some sectors represents a much larger rate of growth than it would for other sectors” (Gray *et al.*, 1979, pp. 68, 72, respectively).

Consider output multipliers; again, the principles are the same for all the other possible multipliers. The idea begins with the multiplier *matrix* $\mathbf{M}(o) = \mathbf{L}$. Row sums, $\mathbf{M}(o)\mathbf{i} = \mathbf{L}\mathbf{i}$, indicate output effects in each sector when final demand for each sector increases by \$1.00. This is generally considered an unlikely scenario; an obvious variation is to posit an unequal increase in final demand across sectors. For example, instead of $\mathbf{L}\mathbf{i}$ one could use $\mathbf{L}\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle\mathbf{i}$, where $\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle$ is a diagonal matrix showing each sector’s final demand as a *proportion* of total final demand, $f_j / \sum_j f_j$; that is, a measure of relative sector size (or importance). (Base-year output proportions, $x_j / \sum_j x_j$, could also be used.) Element (i, j) in the matrix $\mathbf{L}\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle$ shows the effect on sector i output of a $\$(f_j / \sum_j f_j)$ increase in j ’s final demand. Then $\mathbf{L}\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle\mathbf{i}$ shows the multiplier effect on each sector’s output of a \$1 final-demand increase distributed across sectors according to their proportion of total final demand.

Another possibility is to use equal percentage, not absolute, demand increases across sectors. This is the “growth equalization.” For example, elements of the column vector $[\mathbf{M}(o)](0.01)\mathbf{f} = (0.01)\mathbf{L}\mathbf{f}$ indicate output effects in each sector when final demand for each sector increases by one percent, and $(0.01)\mathbf{i}'\mathbf{L}\mathbf{f} = (0.01)[\mathbf{m}(o)]\mathbf{f}$ indicates the economy-wide total output generated. We illustrate with the same three-sector figures.

²⁰ Since $(\mathbf{L} - \mathbf{I}) = \mathbf{L}(\mathbf{I} - \mathbf{L}^{-1}) = \mathbf{L}\mathbf{A}$ or $(\mathbf{L} - \mathbf{I}) = (\mathbf{I} - \mathbf{L}^{-1})\mathbf{L} = \mathbf{A}\mathbf{L}$, these modified multipliers could also be found as $\mathbf{i}'\mathbf{A}\mathbf{L}$ or $\mathbf{i}'\mathbf{L}\mathbf{A}$ (see de Mesnard, 2002, or Dietzenbacher, 2005).

For the example,

$$\mathbf{f} = \begin{bmatrix} 300 \\ 1300 \\ 150 \end{bmatrix} \text{ and } \langle \mathbf{f} \langle \mathbf{i}' \mathbf{f} \rangle^{-1} \rangle = \left[f_j / \sum_j f_j \right] = \begin{bmatrix} 0.1714 & 0 & 0 \\ 0 & 0.7429 & 0 \\ 0 & 0 & 0.0857 \end{bmatrix}$$

In this case,

$$\mathbf{L} \langle \mathbf{f} \langle \mathbf{i}' \mathbf{f} \rangle^{-1} \rangle = \begin{bmatrix} 0.2340 & 0.3159 & 0.0215 \\ 0.0904 & 1.0015 & 0.0510 \\ 0.0977 & 0.3633 & 0.1104 \end{bmatrix} \text{ and } [\mathbf{L} \langle \mathbf{f} \langle \mathbf{i}' \mathbf{f} \rangle^{-1} \rangle] \mathbf{i} = \begin{bmatrix} 0.5714 \\ 1.1429 \\ 0.5714 \end{bmatrix}$$

Using a one percent increase for the growth equalization illustration,

$$\mathbf{L} \langle (0.01) \mathbf{f} \rangle = \begin{bmatrix} 4.0953 & 5.5284 & 0.3764 \\ 1.5820 & 17.5250 & 0.8930 \\ 1.7095 & 6.3576 & 1.9328 \end{bmatrix}$$

and

$$\mathbf{i}' \mathbf{L} \langle (0.01) \mathbf{f} \rangle = [7.3868 \quad 29.4110 \quad 3.2022]$$

Recall that for this example the simple output multipliers were

$$\mathbf{m}(o) = \mathbf{i}' \mathbf{L} = [2.4623 \quad 2.2624 \quad 2.1348]$$

and we see that the relative importance of the sectors is altered (now it is final demand for sector 2 that is the most stimulative; previously – in $\mathbf{m}(o)$ – it was sector 1).

Another Kind of Net Multiplier Standard input–output multipliers (Tables 6.3 and 6.4) are designed to be used with (multiplied by) final demand. Oosterhaven and Stelder (2002a, 2002b) have observed that in the real world, “practitioners” sometimes (perhaps often) use them incorrectly, to multiply total sectoral output (or value added or employment). So they propose *net* multipliers (the terminology could be confusing; these are not multipliers in a net model, as in section 6.5.2). Essentially, they simply convert a standard multiplier so that it can be used in conjunction with total outputs. For example, their Type I *net* output multipliers are $\mathbf{i}' \mathbf{L} \hat{\mathbf{f}}_c$, where $\mathbf{f}_c = [f_j/x_j]$; in their terms, f_j/x_j is the fraction of j ’s output that may “rightfully be considered exogenous” (Oosterhaven and Stelder, 2002a, p. 536). Specifically, they “decompose” $\mathbf{i}' \mathbf{L} \mathbf{f}$ as follows:

$$\mathbf{i}' \mathbf{L} \mathbf{f} = \mathbf{m}(o) \mathbf{f} = \mathbf{m}(o) \hat{\mathbf{f}} \mathbf{i} = \mathbf{m}(o) \hat{\mathbf{f}} \hat{\mathbf{x}}^{-1} \hat{\mathbf{x}} \mathbf{i} = \mathbf{m}(o) \hat{\mathbf{f}}_c \mathbf{x} = \mathbf{i}' \mathbf{L} \hat{\mathbf{f}}_c \mathbf{x}$$

The *net* multiplier *matrix* is thus $\mathbf{L} \hat{\mathbf{f}}_c$ and the associated *vector* of economy-wide multipliers is $\mathbf{i}' \mathbf{L} \hat{\mathbf{f}}_c = \mathbf{m}(o) \hat{\mathbf{f}}_c$. Other multipliers can be similarly modified.

This work generated considerable discussion and a lengthy and elaborate exchange (de Mesnard, 2002, 2007a, 2007b; Dietzenbacher, 2005, Oosterhaven, 2007), with a variety of interpretations and alternative terminology. In the end, “net contribution”

or “net backward linkage” indicators were suggested as a more appropriate label than “multiplier.” We will return to an aspect of this in Chapter 12 on linkage measures in input–output models.

6.6 Multipliers and Elasticities

6.6.1 Output Elasticity

Another approach to compensating for differences in industry size is one step further from simply considering percentage increases in final demand (as above, in growth equalized multipliers). The idea is to measure both the stimulus *and* its effect in percentage terms – in this case the percentage change in total output (or income or employment, etc.) due to a percentage change in a given industry’s final demand. (See, for example, Mattas and Shrestha, 1991 or Ciobanu, Mattas and Psaltopoulos, 2004.) These (percentage change)/(percentage change) measures are “elasticities” in economics terms.

In particular, consider a one percent change in f_j only, so $(\Delta \mathbf{f})' = [0, \dots, (0.01)f_j, \dots, 0]$. Then $\Delta \mathbf{x} = \mathbf{L}\Delta \mathbf{f} = \begin{bmatrix} l_{1j} \\ \vdots \\ l_{nj} \end{bmatrix} (0.01)f_j$. The economy-wide output change

is $\mathbf{i}'\Delta \mathbf{x} = \mathbf{i}' \begin{bmatrix} l_{1j} \\ \vdots \\ l_{nj} \end{bmatrix} (0.01)f_j = \mathbf{m}(o)_j (0.01)f_j$. This percentage change in total output (across all industries) that is generated by $(0.01)f_j$ has been labeled the *output elasticity* of industry j (oe_j) and is defined as

$$oe_j = 100 \times (\mathbf{i}'\Delta \mathbf{x}/\mathbf{i}'\mathbf{x}) = 100 \times \mathbf{m}(o)_j [(0.01)f_j/\mathbf{i}'\mathbf{x}] = \mathbf{m}(o)_j [f_j/\mathbf{i}'\mathbf{x}]$$

(It would be more precise to call this an *output-to-final demand* elasticity, to distinguish it from other elasticities, below.)

Modification of any of the other multipliers in section 6.2.2 – through multiplication by $[f_j/\mathbf{i}'\mathbf{x}]$ – produces exactly parallel results, giving income, employment, etc., elasticities to final demand. Note that these are very similar to the “growth-equalized” multipliers above; in that case, the modification was produced by $\left[f_j / \sum_j f_j \right]$ while here it is $\left[f_j / \sum_j x_j \right]$.

6.6.2 Output-to-Output Multipliers and Elasticities

Direct Effects Starting with $z_{ij} = a_{ij}x_j$, consider the direct effect of an exogenous change in industry j ’s output (Δx_j) – $\Delta x_j \rightarrow \Delta z_{ij} = a_{ij}\Delta x_j$. This Δz_{ij} represents new i output *directly* required by j , so $\Delta x_i = \Delta z_{ij}$, and thus $\Delta x_i = a_{ij}\Delta x_j$ or $\Delta x_i/\Delta x_j = a_{ij}$. Now consider a one percent increase in j ’s output $\Delta x_j = (0.01)x_j$; this means

$\Delta x_i = (0.01)a_{ij}x_j$. So the (i, j) th element of the matrix $(0.01)\mathbf{A}\hat{\mathbf{x}}$ measures the direct effect of j 's one percent increase in output on industry i . Expressed as a percentage of i 's output, we have $100(\Delta x_i/x_i) = 100(0.01)a_{ij}x_j/x_i = a_{ij}x_j/x_i$. And in matrix form, this is the (i, j) th element of the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$, showing the *direct* effect on industry i 's output (percentage change) resulting from a one percent change in industry j 's output. This is a *direct output-to-output* elasticity. We will meet the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$ again in Chapter 12, where we explore supply-side input–output models.

Total Effects Elements of the Leontief inverse matrix translate final demand changes into *total* output changes – $\Delta x_i = l_{ij}\Delta f_j$ and $l_{ij} = \Delta x_i/\Delta f_j$. These encompass direct and indirect effects, and they are at the heart of the multipliers explored in previous sections in this chapter. Again, it would be slightly cumbersome but completely accurate to call l_{ij} an *output-to-final-demand multiplier*. Consider l_{jj} , the on-diagonal element in the j th column of \mathbf{L} – $l_{jj} = \Delta x_j/\Delta f_j$ or $\Delta x_j = l_{jj}\Delta f_j$. Define l_{ij}^* as l_{ij}/l_{jj} ; then

$$l_{ij}^* = l_{ij}/l_{jj} = [\Delta x_i/\Delta f_j]/[\Delta x_j/\Delta f_j] = \Delta x_i/\Delta x_j$$

or $\Delta x_i = l_{ij}^*\Delta x_j$. Thus, l_{ij}^* could be (and has been) viewed as a *total output-to-output* multiplier.

The matrix of these multipliers, $\mathbf{L}^* = [l_{ij}^*]$, is created by dividing each element in a column of \mathbf{L} by the on-diagonal element for that column – $\mathbf{L}^* = \mathbf{L}(\hat{\mathbf{L}})^{-1}$ (as usual, $\hat{\mathbf{L}}$ is a diagonal matrix created from the on-diagonal elements in \mathbf{L}). Then each of the elements in column j of \mathbf{L}^* indicates the amount of change in industry i output (the row label) that would be required if the *output* of industry j were increased by one dollar.²¹

Suppose, then, that industry j is projected to increase its output to some new amount, \bar{x}_j . Postmultiplication of \mathbf{L}^* by a vector, $\bar{\mathbf{x}}$, with \bar{x}_j as its j th element and zeros elsewhere, will generate a vector of total new outputs, \mathbf{x}^* , necessary from each industry in the economy because of the exogenously determined output in industry j . That is,

$$\mathbf{x}^* = \mathbf{L}^*\bar{\mathbf{x}} \quad (6.50)$$

We return to this matrix in Chapter 13 in the context of “mixed” input–output models in which final demands (for some industries) and gross outputs (for the other industries) are specified exogenously.

Moving to elasticity terms, the (i, j) th element of $(0.01)\mathbf{L}\hat{\mathbf{x}}$ gives the (total) new output in industry i caused by a one-percent output increase in industry j . So, exactly parallel to the direct elasticity case, above, the (i, j) th element of $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ gives the percent increase in industry i total output due to an initial exogenous one percent increase in industry j output – the “direct and indirect output elasticity of industry i with respect to the output

²¹ This is equivalent to the “total flow” approach of Szyrmer (for example, Szyrmer, 1992). He makes a case for the unsuitability of the usual output multipliers (from the standard demand-driven input–output model) for a wide variety of real-world impact studies. Some analysts argue that the *initial* exogenous one-dollar stimulus should be removed from the “total effect” calculation. As was seen above (section 6.5.3), this can be accomplished by replacing \mathbf{L} by $(\mathbf{L} - \mathbf{I})$. The interested reader should see de Mesnard (2002) and Dietzenbacher (2005) for details.

in industry j " (Dietzenbacher, 2005, p. 426). We will also meet this matrix, $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$, again in Chapter 12 in the discussion of supply-side input–output models.

6.7 Multiplier Decompositions

A number of approaches have been suggested for analyzing the economic “structure” that is portrayed in input–output data. Multiplier decompositions are a prominent part of this research, and we explore two of these in this section.²²

6.7.1 Fundamentals

We start with the fundamental input–output accounting relationship

$$\underset{(n \times 1)}{\mathbf{x}} = \underset{(n \times n)}{\mathbf{A}} \underset{(n \times 1)}{\mathbf{x}} + \underset{(n \times 1)}{\mathbf{f}} \quad (6.51)$$

from which $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{L}\mathbf{f}$. We now introduce some algebra that initially appears unmotivated but it will soon be clear what is accomplished. Given some $\tilde{\mathbf{A}}_{(n \times n)}$, adding and subtracting $\tilde{\mathbf{A}}\mathbf{x}$ to (6.51) and rearranging produces

$$\mathbf{x} = \mathbf{A}\mathbf{x} - \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{A}}\mathbf{x} + \mathbf{f} \Rightarrow (\mathbf{I} - \tilde{\mathbf{A}})\mathbf{x} = (\mathbf{A} - \tilde{\mathbf{A}})\mathbf{x} + \mathbf{f} \quad (6.52)$$

and, solving²³ for \mathbf{x} ,

$$\mathbf{x} = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}})\mathbf{x} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f}$$

Let $\mathbf{A}^* = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}})$; then this is

$$\mathbf{x} = \mathbf{A}^*\mathbf{x} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \quad (6.53)$$

Next, premultiply both sides of (6.53) by \mathbf{A}^*

$$\mathbf{A}^*\mathbf{x} = (\mathbf{A}^*)^2\mathbf{x} + \mathbf{A}^*(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \quad (6.54)$$

and substitute this for $\mathbf{A}^*\mathbf{x}$ in the right-hand side of (6.53)

$$\mathbf{x} = (\mathbf{A}^*)^2\mathbf{x} + \mathbf{A}^*(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} = (\mathbf{A}^*)^2\mathbf{x} + (\mathbf{I} + \mathbf{A}^*)(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \quad (6.55)$$

Again, solving for \mathbf{x} ,

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^2]^{-1}}_{\mathbf{M}_3} \underbrace{(\mathbf{I} + \mathbf{A}^*)}_{\mathbf{M}_2} \underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1} \mathbf{f} \quad (6.56)$$

In this way the usual Leontief inverse (multiplier) matrix, $(\mathbf{I} - \mathbf{A})^{-1}$, has been decomposed into the product of three matrices.

²² For an overview of these and several others, see Sonis and Hewings (1988) or additional references noted in section 14.2, below.

²³ Here and throughout we assume nonsingularity of the matrices whose inverses are shown.

This algebra can be continued. Premultiply both sides of (6.55) by A^* ,

$$A^*x = (A^*)^3x + [A^* + (A^*)^2](I - \tilde{A})^{-1}f \quad (6.57)$$

and, again, substitute for A^*x in the right-hand side of (6.53)

$$x = (A^*)^3x + [I + A^* + (A^*)^2](I - \tilde{A})^{-1}f \quad (6.58)$$

Solving for x , we now find

$$x = \underbrace{[I - (A^*)^3]^{-1}}_{M_3} \underbrace{[I + A^* + (A^*)^2]}_{M_2} \underbrace{(I - \tilde{A})^{-1}f}_{M_1} \quad (6.59)$$

[Compare with the results in (6.56).]

In the context of social accounting matrices (Chapter 11), where much of the fundamental work on multiplier decompositions originated, M_1 is said to capture a “transfer” effect, M_2 embodies “open-loop” effects and M_3 contains “closed-loop” effects. (For example, see Pyatt and Round, 1979.) The logic of these labels will be clear in the interregional context, below.

These iterations can continue any number of times. After k steps, the parallel to (6.58) is

$$x = (A^*)^kx + [I + A^* + (A^*)^2 + \cdots + (A^*)^{k-1}](I - \tilde{A})^{-1}f \quad (6.60)$$

and the parallel to (6.59) is

$$x = \underbrace{[I - (A^*)^k]^{-1}}_{M_3} \underbrace{[I + A^* + (A^*)^2 + \cdots + (A^*)^{k-1}]}_{M_2} \underbrace{(I - \tilde{A})^{-1}f}_{M_1} \quad (6.61)$$

6.7.2 Decompositions in an Interregional Context

For a two-region interregional model (section 3.3) the input–output accounting relationship $x = Ax + f$ becomes

$$\begin{bmatrix} x^r \\ x^s \end{bmatrix} = \begin{bmatrix} A^{rr} & A^{rs} \\ A^{sr} & A^{ss} \end{bmatrix} \begin{bmatrix} x^r \\ x^s \end{bmatrix} + \begin{bmatrix} f^r \\ f^s \end{bmatrix}$$

With a view toward decompositions, we can isolate the intraregional and interregional elements in A ; let

$$A = \begin{bmatrix} A^{rr} & A^{rs} \\ A^{sr} & A^{ss} \end{bmatrix} = \begin{bmatrix} A^{rr} & 0 \\ 0 & A^{ss} \end{bmatrix} + \begin{bmatrix} 0 & A^{rs} \\ A^{sr} & 0 \end{bmatrix}$$

Define $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{ss} \end{bmatrix}$ from which $(\mathbf{I} - \tilde{\mathbf{A}}) = \begin{bmatrix} \mathbf{I} - \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{A}^{ss} \end{bmatrix}$. Then, using the decomposition in (6.56), for example,

$$\mathbf{M}_1 = (\mathbf{I} - \tilde{\mathbf{A}})^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \end{bmatrix}$$

(from the rule that the inverse for a block-diagonal matrix is made up of the inverses of the matrices on the main diagonal). Also,

$$\begin{aligned} \mathbf{A}^* &= (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}}) \\ &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{A}^{rs} \\ (\mathbf{I} - \mathbf{A}^{ss})^{-1} \mathbf{A}^{sr} & \mathbf{0} \end{bmatrix} \end{aligned}$$

and so, again from (6.56),

$$\mathbf{M}_2 = \mathbf{I} + \mathbf{A}^* = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{A}^{rs} \\ (\mathbf{I} - \mathbf{A}^{ss})^{-1} \mathbf{A}^{sr} & \mathbf{I} \end{bmatrix}$$

Finally, from straightforward matrix multiplication,

$$(\mathbf{A}^*)^2 = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{A}^{rs} (\mathbf{I} - \mathbf{A}^{ss})^{-1} \mathbf{A}^{sr} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \mathbf{A}^{sr} (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{A}^{rs} \end{bmatrix}$$

and so

$$\begin{aligned} \mathbf{M}_3 &= [\mathbf{I} - (\mathbf{A}^*)^2]^{-1} = \\ &= \begin{bmatrix} [\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{A}^{rs} (\mathbf{I} - \mathbf{A}^{ss})^{-1} \mathbf{A}^{sr}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathbf{I} - (\mathbf{I} - \mathbf{A}^{ss})^{-1} \mathbf{A}^{sr} (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{A}^{rs}]^{-1} \end{bmatrix} \end{aligned}$$

(again from the rule for the inverse of a block-diagonal matrix).

In terms of intra- and interregional effects, the matrices in \mathbf{M}_1 are seen to capture *intraregional* (Leontief inverse or “transfer”) effects, those in \mathbf{M}_2 contain *interregional spillover* (“open-loop”) effects, and the matrices in \mathbf{M}_3 record *interregional feedback* (“closed-loop”) effects (Round, 1985, 2001; Dietzenbacher, 2002).²⁴ As usual, define

$$\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1} \quad \text{and} \quad \mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1}$$

²⁴ There have been other definitions of these various effects in the input-output literature, beginning perhaps with Miller (1966, 1969) but also including, among others, Yamada and Ihara (1969), Round (1985, 2001), or Sonis and Hewings (2001).

These are the intraregional effects in each region (\mathbf{M}_1). The two spillover matrices in \mathbf{M}_2 may be represented as

$$\mathbf{S}^{rs} = \mathbf{L}^{rr} \mathbf{A}^{rs} \text{ and } \mathbf{S}^{sr} = \mathbf{L}^{ss} \mathbf{A}^{sr}$$

and the two feedback matrices in \mathbf{M}_3 can be defined as

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{L}^{rr} \mathbf{A}^{rs} \mathbf{L}^{ss} \mathbf{A}^{sr}]^{-1} \text{ and } \mathbf{F}^{ss} = [\mathbf{I} - \mathbf{L}^{ss} \mathbf{A}^{sr} \mathbf{L}^{rr} \mathbf{A}^{rs}]^{-1}$$

or

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{S}^{rs} \mathbf{S}^{sr}]^{-1} \text{ and } \mathbf{F}^{ss} = [\mathbf{I} - \mathbf{S}^{sr} \mathbf{S}^{rs}]^{-1}$$

Therefore, in the two-region interregional context, $\mathbf{x} = \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1 \mathbf{f}$ becomes

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{S}^{rs} \\ \mathbf{S}^{sr} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \quad (6.62)$$

or, carrying out the multiplications,

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{rr} \mathbf{L}^{rr} & \mathbf{F}^{rr} \mathbf{S}^{rs} \mathbf{L}^{ss} \\ \mathbf{F}^{ss} \mathbf{S}^{sr} \mathbf{L}^{rr} & \mathbf{F}^{ss} \mathbf{L}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \quad (6.63)$$

6.7.3 Stone's Additive Decomposition

An alternative decomposition isolates *net* effects. Starting with the multiplicative result in (6.56) [or (6.59), or (6.61)], namely $\mathbf{x} = \mathbf{M} \mathbf{f}$, where $\mathbf{M} = \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1$, Stone (1985) proposed the additive form

$$\mathbf{M} = \mathbf{I} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}$$

(This is easily seen to be true by simply carrying out the algebra on the right-hand side.) Therefore,

$$\mathbf{x} = \mathbf{M} \mathbf{f} = \mathbf{I} \mathbf{f} + \underbrace{(\mathbf{M}_1 - \mathbf{I}) \mathbf{f}}_{\tilde{\mathbf{M}}_1} + \underbrace{(\mathbf{M}_2 - \mathbf{I}) \mathbf{M}_1 \mathbf{f}}_{\tilde{\mathbf{M}}_2} + \underbrace{(\mathbf{M}_3 - \mathbf{I}) \mathbf{M}_2 \mathbf{M}_1 \mathbf{f}}_{\tilde{\mathbf{M}}_3} \quad (6.64)$$

To paraphrase Stone (p. 162) – in the context of an interregional model – we start with a matrix of initial injections, $\mathbf{I} \mathbf{f}$. The second term ($\tilde{\mathbf{M}}_1 \mathbf{f}$) adds on the *net* intraregional effects captured in \mathbf{M}_1 . Next (in $\tilde{\mathbf{M}}_2 \mathbf{f}$) we add in the net interregional spillover effects in \mathbf{M}_2 . Finally, the fourth term ($\tilde{\mathbf{M}}_3 \mathbf{f}$) captures the net interregional feedback effects in

\mathbf{M}_3 . In the two-region example, these are

$$\begin{aligned}\tilde{\mathbf{M}}_1 &= \mathbf{M}_1 - \mathbf{I} = \begin{bmatrix} \mathbf{L}^{rr} - \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} - \mathbf{I} \end{bmatrix} \\ \tilde{\mathbf{M}}_2 &= (\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{S}^{rs} \\ \mathbf{S}^{sr} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{0} \end{bmatrix} \\ \tilde{\mathbf{M}}_3 &= (\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} \mathbf{F}^{rr}\mathbf{L}^{rr} - \mathbf{L}^{rr} & \mathbf{F}^{rr}\mathbf{S}^{rs}\mathbf{L}^{ss} - \mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{F}^{ss}\mathbf{S}^{sr}\mathbf{L}^{rr} - \mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{F}^{ss}\mathbf{L}^{ss} - \mathbf{L}^{ss} \end{bmatrix}\end{aligned}$$

While these appear (and are) increasingly complex, they serve to disentangle the complex net of intraregional, spillover, and feedback effects.

6.7.4 A Note on Interregional Feedbacks

Interregional feedback effects in a two-region input-output model were explored in section 3.3.2. They were defined early on (Miller 1966, 1969) for the specific scenario of a change in final demand in region r only – so $\Delta \mathbf{f}^r \neq \mathbf{0}$ and $\Delta \mathbf{f}^s = \mathbf{0}$. Then a measure of the interregional feedback effect is found as the difference between the output change in region r that would be generated by the complete two-region model and the output change in region r that would be calculated from a single-region model. These outputs are

$$\Delta \mathbf{x}_T^r = [(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1} \Delta \mathbf{f}^r \text{ and } \Delta \mathbf{x}_S^r = (\mathbf{I} - \mathbf{A}^{rr})^{-1} \Delta \mathbf{f}^r$$

(with subscripts indicating “two-region” and “single-region” models, respectively). Consider the inverse matrix in $\Delta \mathbf{x}_T^r$, $[(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1}$

1. Factoring out $(\mathbf{I} - \mathbf{A}^{rr})$ gives

$$\{(\mathbf{I} - \mathbf{A}^{rr})[\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]\}^{-1}$$

2. Using the rule that $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$, we have

$$[\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1}(\mathbf{I} - \mathbf{A}^{rr})^{-1}$$

Using $\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1}$ and $\mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1}$, we have

$$\Delta \mathbf{x}_T^r = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}\mathbf{L}^{rr}\Delta \mathbf{f}^r \text{ and } \Delta \mathbf{x}_S^r = \mathbf{L}^{rr}\Delta \mathbf{f}^r$$